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The Enumeration of c-Nets Via Quadrangulations

R. C. MULLIN AND P. J. SCHELLENBERG

Department of Mathematics, University of Waterloo, Waterloo, Ontario, Canada

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ABSTRACT

A counting procedure for simple quadrangulations is established. Using the technique of counting simple quadrangulations together with a one-to-one correspondence between simple quadrangulations and c-nets, the enumeration of c-nets with $i + 1$ vertices and $j + 1$ faces is accomplished.

1. INTRODUCTION

The definition of quadrangulation is cited as given by Brown [1, p. 302]. Let R be a simply-connected closed region in E^2 bounded by a simple closed curve C. A *quadrangulation* is a representation of R as the union of a finite number of disjoint sets called cells, where the cells are of three kinds, vertices, edges, and faces (said to be of dimension 0, 1, and 2, respectively), such that: each vertex is a single point, each edge is an open arc whose ends are distinct vertices and no two edges have the same two vertices as ends, and each face is bounded by the closure of the union of four edges. Two cells of different dimension are *incident* if one is contained in the boundary of the other. Vertices and edges are *external* if they are contained in the closure of the compliment of R. Otherwise they are internal.

A *rooted* quadrangulation is one in which one external vertex is distinguished as the root vertex and one external edge incident with the root vertex is distinguished as root edge. A quadrangulation is of type $[n, m]$ if it has n internal vertices and $m + 4$ external vertices. Throughout this paper, the word quadrangulation(s) will refer to *rooted* quadrangulation(s). Two rooted quadrangulations, Q and Q^*, are *isomorphic* if there exists a biunique mapping, f, of the cells of Q onto the cells of Q^* which preserves dimension and rooting and both f and f^{-1} preserve incidence. This paper is concerned with the enumeration of isomorphism classes of rooted simple quadrangulations as defined below.

Any (open) region in R whose boundary is the closure of the union of four edges is called a *quad.* If every quad of the quadrangulation R is a face, then R is said to be a *simple quadrangulation.*

The enumeration of simple quadrangulations of type $[n, m]$ is accomplished in Sections 3 and 4. In Section 5, a correspondence between c-nets (3-connected rooted plane maps) and simple quadrangulations is established. The enumeration of c-nets having $i + 1$ vertices and $j + 1$ faces is accomplished in Section 6 by enumerating simple quadrangulations and then applying the correspondence of Section 5.

2. A Preliminary Lemma

A quad S of R is said to be *maximal* if it is not properly contained in any other quad of R. (Since quads of a quadrangulation are (open) regions, the containment referred to in the above definition is just the usual containment of sets.)

Lemma. *Two maximal quads, C and D, of a quadrangulation R, are either equal or disjoint.*

Proof: C and D are open regions in R. Suppose C and D are not disjoint. Then they intersect in the open region $D \cap C$. For S any region in E^2, let $B(S)$ denote the boundary of the region S.

(i) Suppose $B(D \cap C)$ is the closure of an edge. This implies the existence of a loop in R, which is a contradiction of the definition of quadrangulation.

(ii) Suppose $B(C \cap D)$ is the closure of the union of two edges. This implies multiple joins in R, a contradiction of the definition of quadrangulation.

(iii) Suppose $B(C \cap D)$ is the closure of the union of three edges. Then $(C \cap D) \cup B(C \cap D)$, a subset of R, must also be a quadrangulation. By Euler's formula, it can be proved that no quadrangulation can be bounded by three edges. Thus $B(C \cap D)$ cannot be the closure of three edges.

(iv) Suppose $B(C \cap D)$ is the closure of the union of four edges. Three cases arise.

Case (a): Suppose $B(C \cap D) \subseteq B(D)$. This implies $B(C \cap D) = B(D)$ since both are the closure of the union of four edges. This implies that $D \cap C = D$, i.e., $D \subseteq C$. But D is maximal which implies $D = C$.

CASE (b): Suppose $B(C \cap D)$ contains exactly three edges of $B(D)$. Then $B(C \cap D)$ contains an edge A of $B(C)$ which forms a multiple join when taken together with the fourth edge of $B(D)$. This is a contradiction of the definition of quadrangulation.

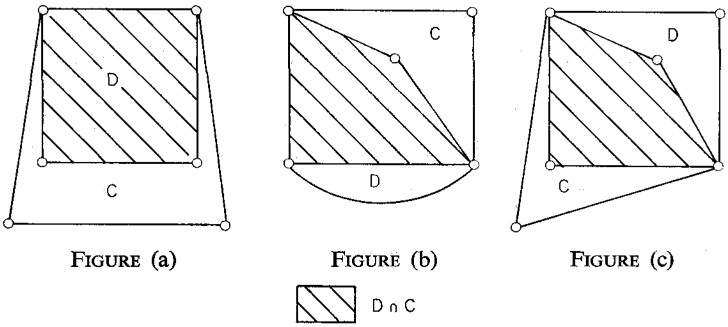

FIGURE (a) FIGURE (b) FIGURE (c)

CASE (c): Suppose $B(D \cap C)$ contains exactly two edges of $B(D)$. If $B(C \cap D)$ contains four edges of $B(C)$ this implies case (a), and if $B(C \cap D)$ contains three edges of $B(C)$ this implies case (b). Suppose $B(C \cap D)$ contains exactly two edges of $B(C)$. This implies that $D \cup C$ is a quad and that $D \subset (D \cup C)$ and $C \subset (D \cup C)$ (proper containment). But D and C are maximal quads; thus we have a contradiction.

(v) Suppose $B(C \cap D)$ is the closure of the union of five edges. Then, $B(C \cap D) \cup (C \cap D)$ must also be a quadrangulation. By Euler's formula, it can be proved that no quadrangulation with five external edges exists, and thus we have a contradiction.

(vi) Suppose $B(C \cap D)$ is the closure of the union of six edges. This is possible only if $B(C \cap D)$ contains three edges of $B(C)$ and three edges

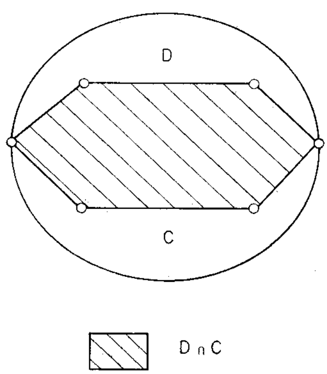

FIGURE (d)

of $B(D)$. But this gives the situation of Figure (d), which implies multiple joins. This contradicts the definition of a quadrangulation. Thus if C and D intersect, they must be equal as seen in (iv) case (a). This completes the lemma.

3. The Enumeration of Simple Quadrangulations of Type $[n, m]$, $m > 0$

A corollary to the lemma of Section 2 is the following: with each quadrangulation, R, there is associated a unique simple quadrangulation, A, obtained by replacing maximal quads in R by faces. Thus it has essentially been proved that, for any quadrangulation R, there exists a unique simple quadrangulation, A, such that R can be obtained from A by replacing faces in A by quadrangulations of type $[n, 0]$. Therefore, the set of all quadrangulations can be constructed from the set of simple quadrangulations by replacing faces in the simple quadrangulation by quadrangulations of type $[n, 0]$.

Let $q_{n,m}$ be the number of simple quadrangulations of type $[n, m]$. By (1, 6.7) or by using Euler's formula it can be shown that

$$q_{n,2m-1} = 0, (m = 1, 2,..., n = 0, 1,...). \tag{3.1}$$

Therefore we define the formal power series generating function,

$$Q = Q(x, y^2) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} q_{n,2m} x^n y^{2m}. \tag{3.11}$$

Let $u_{n,m}$ be the number of quadrangulations of type $[n, m]$. Again, it can be proved that

$$u_{n,2m-1} = 0, (n = 0, 1,..., m = 1, 2,...).$$

Furthermore, only quadrangulations of type $[n, m]$, $m > 0$ are being considered in this section. Thus, we employ the generating function (formal series)

$$U = U(x, y^2) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} u_{n,2m} x^n y^{2m}. \tag{3.2}$$

Brown [1, pp. 305–306] has shown that

$$U = 6 \sum_{m=1}^{\infty} \frac{(3m+2)!}{m!(2m+4)!} [(m+1)(2v^{-3m+4} - 4xv^{-3m-6}) - mv^{-3m-3}] y^{2m} \tag{3.21}$$

where

$$x = uv^2 \tag{3.22}$$

and

$$v = 1 - u. \tag{3.23}$$

Also

$$U^* = U(x, 0) = \frac{1 - 2u}{v^4}, \tag{3.24}$$

where U^* is the figure counting series for quadrangulations of type $[n, 0]$, that is,

$$U^* = \sum_{n=0}^{\infty} u_{n,0} x^n.$$

Suppose there is given a simple quadrangulation of type $[n, m]$. The figure counting series for all quadrangulations obtained from this simple quadrangulation by replacing faces of the simple quadrangulation by quadrangulations of type $[n, 0]$ is

$$x^n y^m (U^*)^{n+m+1}.$$

(Note that the simple quadrangulation of type $[n, m]$ has $n + m + 1$ faces.) If these counting series are summed over all possible simple quadrangulations, the resulting counting series is $U = U(x, y^2)$. (This can be concluded in view of the results of the lemma of Section 2.) Thus

$$\sum_{n=0}^{\infty} \sum_{m=1}^{\infty} q_{n,2m} x^n y^{2m} (U^*)^{n+m+1} = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} u_{n,2m} x^n y^{2m},$$

that is,

$$U^* Q(xU^*, y^2 U^*) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} u_{n,2m} x^n y^{2m}. \tag{3.3}$$

We define X and Y by

$$X = xU^* = \frac{u(1 - 2u)}{v^2} \tag{3.31}$$

and

$$Y^2 = y^2 U^*$$

or

$$y^2 = \frac{Y^2}{U^*} = \frac{Y^2 v^4}{1 - 2u}. \tag{3.32}$$

From (3.21), (3.22), (3.24), and (3.3), (3.31), and (3.32),

$$Q(X, Y^2) = \frac{v^4}{1-2u} 6 \sum_{m=1}^{\infty} \frac{(3m+2)!}{m!(2m+4)!}$$
$$\times [(m+1)\, 2v^{-3m-4}(1-2u) - mv^{-3m-3}] \frac{Y^{2m}v^{4m}}{(1-2u)^m}$$

that is,

$$Q(X, Y^2) = 6 \sum_{m=1}^{\infty} \frac{(3m+2)!}{m!(2m+4)!} \left(\frac{(m+1)\, 2v^m}{(1-2u)^m} - \frac{mv^{m+1}}{(1-2u)^{m+1}} \right) Y^{2m}. \quad (3.4)$$

By making the following substitution

$$\delta = \frac{v}{1-2u}, \quad (3.41)$$

one obtains,

$$Q(X, Y^2) = 6 \sum_{m=1}^{\infty} \frac{(3m+2)!}{m!(2m+4)!} [(m+1)\, 2\delta^m - m\delta^{m+1}]\, Y^{2m}. \quad (3.5)$$

From (3.41) and (3.23),

$$\delta - 1 = \frac{u}{1-2u}. \quad (3.51)$$

From (3.31), (3.41), and (3.51),

$$X = \frac{\delta - 1}{\delta^2},$$

that is,

$$\delta = 1 + X\delta^2.$$

Applying Lagrange's theorem [4, p. 108], one obtains

$$\delta^m = \sum_{n=0}^{\infty} \frac{m(2n+m-1)!}{n!(n+m)!} X^n. \quad (3.7)$$

Substituting (3.7) into (3.5), one obtains

$$Q(X, Y^2) = 6 \sum_{m=1}^{\infty} \frac{(3m+2)!}{m!(2m+4)!} \left[2(m+1) \sum_{n=0}^{\infty} \frac{m(2n+m-1)!}{n!(n+m)!} X^n \right.$$
$$\left. - m \sum_{n=0}^{\infty} \frac{(m+1)(2n+m)!}{n!(n+m+1)!} X^n \right] Y^{2m},$$

that is,

$$Q(X, Y^2) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{6(3m+2)!\,(2n+m-1)!\,(m+1)(m+2)}{(m-1)!\,(2m+4)!\,n!\,(n+m+1)!} X^n Y^{2m}.$$

Thus

$$q_{n,2m} = \frac{(3m+3)!\,(2n+m-1)!}{(m-1)!\,(2m+3)!\,n!\,(n+m+1)!}, \qquad (n = 0, 1,...; m = 1, 2,...),$$

$$q_{n,2m-1} = 0, \qquad (n = 0, 1,..., m = 1, 2,...),$$

which completes the enumeration of rooted simple quadrangulations of type $[n, m]$ for $n = 0, 1,...$ and $m = 1, 2,...$.

4. The Enumeration of Simple Quadrangulations of Type $[n, 0]$

In case of quadrangulations of type $[n, 0]$, the external edges and vertices are considered to bound the external face. By the previous counting argument there is only one simple quadrangulation of type $[n, 0]$, namely $[0, 0]$, but with the interpretation just stated there are many simple quadrangulations of type $[n, 0]$. A counting procedure for these quadrangulations is described in this section. Throughout this section quadrangulation(s) will mean rooted quadrangulation(s) of type $[n, 0]$.

In a quadrangulation of type $[n, 0]$, a *diagonal* is a pair of edges incident upon a common internal vertex of the quadrangulation and with two diagonally opposite external vertices. There is one (rooted) simple quadrangulation of type $[0, 0]$ (Figure (e)) and two of type $[1, 0]$ (Figure (f)). Both simple quadrangulations of type $[1, 0]$ have diagonals and it can be shown that these are the only simple quadrangulations having diagonals.

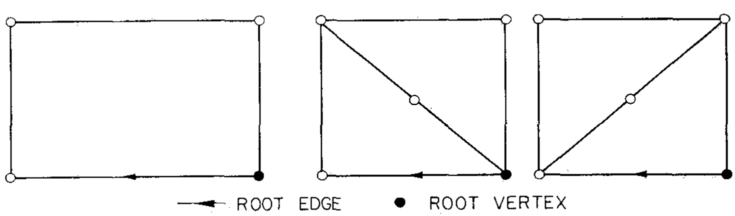

FIGURE (e) FIGURE (f)

Now consider the set of all quadrangulations of type $[n, 0]$ (whose figure counting series is U^* as described in Section 3). These quadrangulations fall into two classes, those with a diagonal and those without a diagonal. U_D^* is defined to be the figure counting series for the former and U_N^* the figure counting series for the latter. Then

$$U^* = U_N^* + U_D^*. \tag{4.1}$$

Recall that $q_{n,0}$ is the number of simple quadrangulations of type $[n, 0]$ and define Q_N the figure counting series for these simple quadrangulations as

$$Q_N = Q_N(x) = \sum_{n=2}^{\infty} q_{n,0}x^n. \tag{4.2}$$

(Note that Q_N counts all simple quadrangulations without diagonals except for the simple quadrangulation $[0, 0]$.) By an argument similar to the argument of Section 3, it can be proved that the set of all quadrangulations without diagonals can be obtained from the set of simple quadrangulations without diagonals by replacing faces of the simple quadrangulations by quadrangulations. Therefore

$$U_N^* = \sum_{n=2}^{\infty} q_{n,0}x^n(U^*)^{n+1} + 1,$$

that is,

$$U_N^* = U^*Q_N(xU^*) + 1. \tag{4.3}$$

From (4.1) and (4.3)

$$U^* = U^*Q_N(xU^*) + 1 + U_D^*. \tag{4.4}$$

Define U_{RD}^* to be the figure counting series for the quadrangulations with a diagonal from the root vertex and U_{NRD}^* to be the figure counting series for the quadrangulations with a diagonal from the non-root vertex. By symmetry,

$$U_{RD}^* = U_{NRD}^* = \frac{U_D^*}{2}. \tag{4.5}$$

Each quadrangulation with a diagonal from the root vertex can be constructed from an arbitrary quadrangulation (all of which are enumerated by U^*) and a quadrangulation with no diagonal from the root vertex (all of which are enumerated by $U_N^* + U_{NRD}^*$) by identifying the root edges of the two quadrangulations and by identifying the external edges incident with the non-root-vertex ends of the root edges of each quadrangulation (Figure (g)). The root vertex of the resultant quadrangulation is the vertex obtained by identifying the root vertices of the two component quadrangulations and the root edge is the edge of the quadrangulation without a diagonal from the root vertex incident with the root vertex.

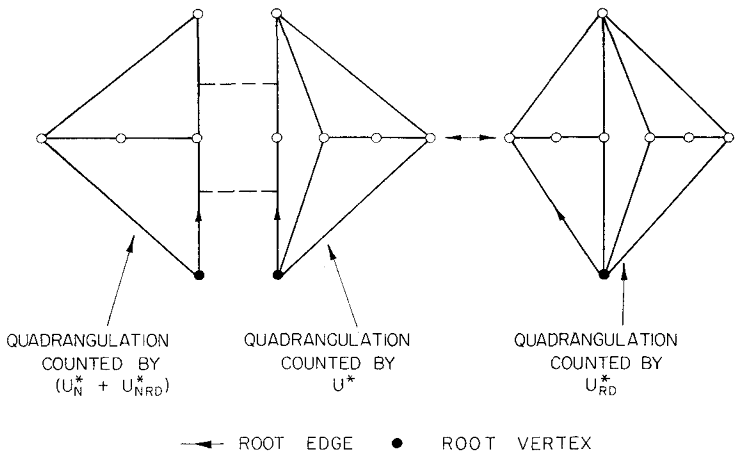

FIGURE (g)

This procedure leads to the functional equation

$$U_{RD}^* = x(U_N^* + U_{NRD}^*)\, U^*. \tag{4.51}$$

Substituting (4.5) in (4.51),

$$\frac{U_D^*}{2} = x\left(U_N^* + \frac{U_D^*}{2}\right) U^*,$$

that is,

$$U_D = \frac{2xU_N^* U^*}{1 - xU^*}. \tag{4.52}$$

Substituting (4.3) into (4.52),

$$U_D^* = \frac{2x(U^*)^2\, Q_N(xU^*) + 2xU^*}{1 - xU^*}. \tag{4.53}$$

Substituting (4.53) into (4.4) and simplifying, one obtains

$$Q_N(xU^*) = \frac{1 - xU^*}{1 + xU^*} - \frac{1}{U^*}. \tag{4.6}$$

Making the substitution

$$X = xU^*,$$

(4.6) becomes

$$Q_N(X) = \frac{1 - X}{1 + X} - \frac{1}{U^*}. \tag{4.62}$$

By substituting from (3.22), (3.23), and (3.24), (4.62), and (4.61) become

$$Q_N(X) = 1 + 2 \sum_{n=1}^{\infty} (-X)^n - \frac{v^4}{1 - 2u} \tag{4.63}$$

and

$$X = \frac{(1 - 2u)}{v^2}, \tag{4.64}$$

respectively. Using (3.23) and (3.41) one can verify that

$$2\delta - 1 = \frac{1}{1 - 2u}, \tag{4.7}$$

and from (3.41) and (4.7) one can obtain

$$\frac{v^4}{1 - 2u} = \frac{\delta^4}{(2\delta - 1)^3}. \tag{4.71}$$

Applying Lagrange's theorem [4, p. 108] to (3.6), one obtains

$$\frac{\delta^4}{(2\delta - 1)^3} = 1 - \sum_{n=2}^{\infty} \left[\sum_{r=0}^{\infty} \frac{(r+2)^2 (r+1)(-2)^r}{2n} \binom{2n+3}{n-r-1} \right] X^n. \tag{4.72}$$

Therefore

$$Q_N(X) = \sum_{n=2}^{\infty} \left[2(-)^n + \sum_{r=0}^{n-1} \frac{(r+2)^2 (r+1)(-2)^r}{2n} \binom{2n+3}{n-r-1} \right] X^n. \tag{4.73}$$

Define the figure counting series for all simple quadrangulations (of type $[n, 0]$) as

$$Q^+ = Q^+(x) = \sum_{n=0}^{\infty} q_{n,0} x^n.$$

Then

$$Q^+ = 1 + 2x + Q_N(x). \tag{4.73}$$

From (4.2) and (4.63), one obtains the result

$$q_{n,0} = 2(-)^n + r_n , \qquad n \geqslant 2, \tag{4.8}$$

where r_n is the coefficient of X^n in $R = R(X)$ defined by

$$R = -\frac{v^4}{1-2u},$$

where X is defined by (4.64). By substituting (3.23), one obtains

$$R = -\frac{(1-u)^4}{1-2u} \tag{4.82}$$

and

$$X = \frac{(1-2u)}{(1-u)^3}. \tag{4.83}$$

In [5, pp. 263–265], Tutte has shown that c_n, the number of c-nets (3-connected rooted planar maps) with n edges, is determined by

$$c_n = 2(-1)^n + s_n , \qquad n \geqslant 4,$$

where s_n is the coefficient of x^{n-1} in $S = S(x)$ defined by

$$27S = \eta(3+\eta)^2, \tag{4.9}$$

where

$$x = \frac{-\eta(3+2\eta)}{(3+\eta)^2}. \tag{4.91}$$

Making the change of variable, $n = k + 2$, one obtains

$$c_{k+2} = 2(-1)^k + t_k , \qquad k \geqslant 2,$$

where t_k is the coefficient of x^k in the expansion of $T = T(x)$. Therefore

$$x^2 T = xS,$$

that is,

$$27T = \frac{27S}{x}. \tag{4.92}$$

Substituting (4.9) and (4.91) in (4.92),

$$27T = \frac{-(3+\eta)^4}{(3+2\eta)}. \tag{4.93}$$

Substituting $\eta = -3\mu$ into (4.93) and (4.91), one obtains

$$T = \frac{-(1-\mu)^4}{(1-2\mu)}, \tag{4.94}$$

where

$$x = \frac{\mu(1-2\mu)}{(1-\mu)^2}. \tag{4.95}$$

Comparing (4.94) and (4.95) with (4.82) and (4.83), respectively, it follows that

$$c_{n+2} = q_{n,0}, \qquad n \geqslant 2.$$

For a partial listing of the $q_{n,0}$ and asymptotic estimates, see [5, pp. 265–266].

5. Relations between c–Nets and Simple Quadrangulations of Type $[n, 0]$

In [1, Section 7], Brown describes a one-to-one correspondence between rooted non-separable planar maps with $n + 2$ edges and rooted quadrangulations of type $[n, 0]$. The following is a brief description of the correspondence. Let M be any rooted non-separable planar map with $n + 2$ edges whose vertex set is W and whose edge set is E. W^* is the set of vertices of the dual map M^*. Every $v \in W^*$ lies in one face, V, of M and v is joined by a new edge to each vertex in $B(V)$. These new edges form a set E^+. The edges of E^+ and the vertices of W and W^* form the quadrangulation of type $[n, 0]$ corresponding to M. The inverse of this operation determines the rooted non-separable planar map with $n + 2$ edges corresponding to a given quadrangulation of type $[n, 0]$. (Note that every quadrangulation is bipartite and so the vertices form two disjoint sets, W and W^*.)

Let G be an arbitrary c-net (3-connected rooted planar map) and let Q be the quadrangulation which corresponds to G. Suppose Q is not a simple quadrangulation of type $[n, 0]$. Since Q is not simple, it contains a maximal quad, L, which is not a face. By the above correspondence, there corresponds to L a subgraph, R, of G with the property that R has only two vertices of attachment with the compliment of R in G; that is, G is 2-connected (but not 3-connected). But this contradicts the fact that G is a c-net. Thus it can be concluded that if G is a c-net with $n + 2$ edges, the quadrangulation of type $[n, 0]$ corresponding to it is simple. This conclusion together with the fact that $c_{n+2} = q_{n,0}$ is sufficient to prove that the given correspondence determines a one-to-one correspondence between c-nets with $n + 2$ edges and simple quadrangulations of type $[n, 0]$.

6. The Enumeration of c–Nets with $i + 1$ Vertices and $j + 1$ Faces

Throughout this section quadrangulation(s) will mean rooted quadrangulation(s) of type $[n, 0]$.

Let f_{ij} be the number of rooted non-separable planar maps with $i + 1$ vertices and $j + 1$ faces and define the generating function, F, as the formal power series

$$F = F(x, y) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} f_{ij} x^i y^j. \tag{6.1}$$

By the correspondence of Section 5, f_{ij} is the number of quadrangulations with $| W | = i + 1$ and $| W^* | = j + 1$, and F is the figure counting series for these quadrangulations. (If X is a set, $| X |$ denotes its cardinality.) F_N, F_{RD}, and F_{NRD} are defined as the figure counting series for quadrangulations without a diagonal, for quadrangulations with a diagonal from the root vertex and for quadrangulations with a diagonal form the non-root vertex, respectively. Then F can be written as the following sum:

$$F = F_N + F_{RD} + F_{NRD}. \tag{6.11}$$

By the same argument used to establish (4.51) one can obtain

$$F_{RD} = (F_N + F_{NRD}) \frac{F}{x}, \tag{6.12}$$

and similarly

$$F_{NRD} = (F_N + F_{RD}) \frac{F}{y}. \tag{6.13}$$

Substituting (6.13) in (6.11) one obtains

$$F = F_N + F_{RD} + (F_N + F_{RD}) \frac{F}{y},$$

that is,

$$F_N + F_{RD} = \frac{F}{1 + \frac{F}{y}}. \tag{6.14}$$

Similarly,

$$F_N + F_{NRD} = \frac{F}{1 + \frac{F}{x}}. \tag{6.15}$$

By summing (6.14) and (6.15) one obtains

$$F_N + F_N + F_{RD} + F_{NRD} = \frac{F}{1 + \frac{F}{y}} + \frac{F}{1 + \frac{F}{x}}.$$

Using (6.11) this becomes

$$F_N = F\left(\frac{1}{1 + \frac{F}{y}} + \frac{1}{1 + \frac{F}{x}} - 1\right). \tag{6.16}$$

Let q_{ij}^* be the number of simple quadrangulations with no diagonal having $|W| = i + 1$ and $|W^*| = j + 1$ and define the generating function Q_N^* as the formal power series

$$Q_N^* = Q_N^*(x, y) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} q_{ij}^* x^i y^j, \tag{6.2}$$

where we define $q_{11} = 0$.

By an argument similar to that of Section 3 it can be proved that each member of the class of all quadrangulations without diagonals can be obtained from a unique member of the class of simple quadrangulations without diagonals by replacing internal faces of the simple quadrangulation by quadrangulations. The counting series for all quadrangulations obtained from a given simple quadrangulation with $|W| = i + 1$ and $|W^*| = j + 1$ is

$$x^i y^j \left(\frac{F}{xy}\right)^{i+j-1} = \left(\frac{F}{y}\right)^i \left(\frac{F}{x}\right)^j \frac{xy}{F}. \tag{6.21}$$

(The given quadrangulation has $i + j - 1$ internal faces.) By summing these counting series for all possible simple quadrangulations one obtains

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} q_{ij}^* x^i y^j \left(\frac{F}{xy}\right)^{i+j-1} = F_N(x, y) - xy,$$

that is,

$$\frac{xy}{F} Q_N^*\left(\frac{F}{y}, \frac{F}{x}\right) = F_N(x, y) - xy. \tag{6.22}$$

Substituting (6.16) in (6.22) one obtains

$$\frac{xy}{F} Q_N^*\left(\frac{F}{y}, \frac{F}{x}\right) = F\left(\frac{1}{1+\frac{F}{y}} + \frac{1}{1+\frac{F}{x}} - 1\right) - xy. \tag{6.23}$$

Let $X = F/y$ and $Y = F/x$. Multiplying through by F/xy and substituting, (6.23) becomes

$$Q_N^*(X, Y) = XY\left(\frac{1}{1+X} + \frac{1}{1+Y} - 1\right) - F. \tag{6.24}$$

(This equation was previous obtained by Tutte from different considerations.)

In [2, p. 572], Brown and Tutte have shown that F is defined by

$$F = uv(1 - u - v), \tag{6.25}$$

where

$$x = u(1 - v)^2$$

and

$$y = v(1 - u)^2.$$

Therefore,

$$X = \frac{F}{y} = \frac{(1 - u - v)}{(1 - u)^2}, \tag{6.3}$$

$$Y = \frac{F}{x} = \frac{v(1 - u - v)}{(1 - v)^2}. \tag{6.31}$$

Let

$$r = \frac{u}{1 - u - v} \tag{6.4}$$

and

$$s = \frac{v}{1 - u - v}. \tag{6.41}$$

Using (6.4) and (6.41), the equations (6.25), (6.3), and (6.31) become

$$F = \frac{rs}{(r + s + 1)^3}, \tag{6.5}$$

$$r = X(s + 1)^2, \tag{6.51}$$

and

$$s = Y(r+1)^2, \tag{6.52}$$

respectively. Using the binomial expansion:

$$\frac{rs}{(r+s+1)^3} = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} (-)^{i+j} \frac{(i+j+2)!}{i!\,j!\,2!} r^{i+1}s^{j+1}. \tag{6.53}$$

Since i and j are non-negative integers, using (6.51), (6.52), and [3, Theorem 12], one obtains

$$r^{i+1}s^{j+1} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{X^m Y^n}{m!\,n!} \times \left[\frac{\partial^{m+n}}{\partial r^m\,\partial s^n}\{r^{i+1}s^{j+1}(r+1)^{2n}(s+1)^{2m}[1-4x(r+1)(s+1)]\right]_{\substack{r=0\\ s=0}},$$

that is,

$$r^{i+1}s^{j+1} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} X^m Y^n \times \left[\binom{2n}{m-i-1}\binom{2m}{n-j-1} - 4\binom{2n-1}{m-i-2}\binom{2m-1}{n-j-2}\right]. \tag{6.54}$$

Substituting (6.54) into (6.53), one obtains

$$F = \frac{rs}{(r+s+1)^3} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} (-)^{i+j} \frac{(i+j+2)!}{i!\,j!\,2!} \times \left\{\binom{2n}{m-i-1}\binom{m}{n-j-1} - 4\binom{2n-1}{m-i-2}\binom{m-1}{n-j-2}\right\} X^m Y^n. \tag{6.55}$$

Thus one has essentially determined Q_N^*, the figure counting series of all simple quadrangulations with no diagonals and having $|W| = m+1$ and $|W^*| = n+1$. But by the correspondence of Section 5 this is also the figure counting series of all c-nets having $m+1$ vertices and $n+1$ faces. Thus

$$q_{1,1}^* = 0,$$

$$q_{1,n}^* = (-1)^{n+1} - \sum_{j=0}^{n-1} (-1)^j \frac{(j+2)!}{2!\,j!}\left[\binom{2}{n-j-1} - 4\binom{1}{n-j-2}\right],$$

n \ m	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0														
2	0	0													
3	0	0	1												
4	0	0	0	4											
5	0	0	0	3	24										
6	0	0	0	0	33	188									
7	0	0	0	0	13	338	1705								
8	0	0	0	0	0	252	3580	16980							
9	0	0	0	0	0	68	3740	39525	180670						
10	0	0	0	0	0	0	1938	51300	452865	2020120					
11	0	0	0	0	0	0	399	38076	685419	5354832	23478426				
12	0	0	0	0	0	0	0	15180	646415	9095856	65022840	281481880			
13	0	0	0	0	0	0	0	2530	373175	10215450	120872850	807560625	3461873536		
14	0	0	0	0	0	0	0	0	121095	7580040	155282400	1614234960	10224817515	43494961412	
15	0	0	0	0	0	0	0	0	16965	3585270	138770307	2308636872	21697730849	131631305614	556461655783

for $n = 2, 3, \ldots$,

$$q^*_{m,1} = (-1)^{m+1} - \sum_{i=0}^{m-1} (-)^i \frac{(i+2)!}{2!\, i!} \left[\binom{2}{m-i-1} - 4\binom{1}{m-i-2}\right],$$

for $m = 2, 3, \ldots$, and

$$q^*_{m,n} = -\sum_{i=0}^{m-1} \sum_{j=0}^{n-1} (-)^{i+j} \frac{(i+j+2)!}{2!\, i!\, j!}$$
$$\times \left[\binom{2n}{m-i-1}\binom{2m}{n-j-1} - 4\binom{2n-1}{m-i-2}\binom{m-1}{n-j-2}\right],$$

for $m = 2, 3, \ldots$ and $n = 2, 3, \ldots$.

A table of values for $q^*_{m,n}$ is given.

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